

# THE $J$ -FLOW ON KÄHLER SURFACES: A BOUNDARY CASE<sup>1</sup>

HAO FANG, MIJIA LAI, JIAN SONG, AND BEN WEINKOVE

**ABSTRACT.** We study the  $J$ -flow on Kähler surfaces when the Kähler class lies on the boundary of the open cone for which global smooth convergence holds, and satisfies a nonnegativity condition. We obtain a  $C^0$  estimate and show that the  $J$ -flow converges smoothly to a singular Kähler metric away from a finite number of curves of negative self-intersection on the surface. We discuss an application to the Mabuchi energy functional on Kähler surfaces with ample canonical bundle.

## 1. INTRODUCTION

The  $J$ -flow is a parabolic flow on Kähler manifolds with two Kähler classes. It was defined by Donaldson [D1] in the setting of moment maps and by Chen [Ch1] as the gradient flow of the  $\mathcal{J}$ -functional appearing in his formula for the Mabuchi energy [M].

The  $J$ -flow is defined as follows. Let  $X$  be a compact Kähler manifold with two Kähler metrics  $\omega$  and  $\chi$  in different Kähler classes  $[\omega]$  and  $[\chi]$  respectively. Let  $\mathcal{P}_\chi$  be the space of smooth  $\chi$ -plurisubharmonic functions on  $X$ . Namely,

$$\mathcal{P}_\chi = \{\varphi \mid \chi_\varphi := \chi + dd^c \varphi > 0\}.$$

Then the  $J$ -flow is a flow defined in  $\mathcal{P}_\chi$  by

$$(1.1) \quad \frac{\partial}{\partial t} \varphi = c - \frac{n\chi_\varphi^{n-1} \wedge \omega}{\chi_\varphi^n}, \quad \varphi(0) = \varphi_0 \in \mathcal{P}_\chi,$$

where  $c$  is the topological constant given by

$$c = \frac{n[\chi]^{n-1} \cdot [\omega]}{[\chi]^n}.$$

A stationary point of (1.1) gives a critical Kähler metric  $\tilde{\chi} \in [\chi]$  satisfying

$$(1.2) \quad c\tilde{\chi}^n = n\tilde{\chi}^{n-1} \wedge \omega.$$

Donaldson noted that a smooth critical metric exists only if the cohomological condition  $[c\chi - \omega] > 0$  holds [D1]. In complex dimension 2, Chen [Ch1] showed that this necessary condition is sufficient for the existence of a smooth critical metric, by observing that in this case (1.2) is equivalent to the complex Monge-Ampère equation solved by Yau [Y1] (see (2.2) below). Chen also established the long time existence for the  $J$ -flow (1.1) with any initial data [Ch2]. The fourth-named author [W1, W2] showed that the  $J$ -flow converges to a critical metric if the cohomological condition  $[c\chi - (n-1)\omega] > 0$  holds. In particular, if  $X$  is a Kähler surface, a necessary and sufficient condition for convergence of the flow to a smooth critical metric is Donaldson's cohomological condition  $[c\chi - \omega] > 0$ .

<sup>1</sup>Research supported in part by NSF grants DMS-1008249, DMS-08047524 and DMS-1105373.

In [SW], the third and fourth-named authors found a necessary and sufficient condition for the convergence of the  $J$ -flow in higher dimensions, which we now explain. Define

$$(1.3) \quad \mathcal{C}_\omega := \{ [\chi] > 0 \mid \exists \chi' \in [\chi] \text{ such that } c\chi'^{n-1} - (n-1)\chi'^{n-2} \wedge \omega > 0 \}.$$

Then it was shown in [SW] that the  $J$ -flow (1.1) converges smoothly to the critical metric solving (1.2) if and only if  $[\chi] \in \mathcal{C}_\omega$ .

In [FLM, FL1], the  $J$ -flow has been generalized to the general inverse  $\sigma_k$  flow. An analogous necessary and sufficient condition is found to ensure the smooth convergence of the flow.

The behavior of the  $J$ -flow in the case when the condition  $[\chi] \in \mathcal{C}_\omega$  does *not* hold is still largely open. However, recent progress has been made by the first and second named authors [FL2] in the case of a family of Kähler manifolds satisfying the Calabi symmetry condition. It was shown (in the more general case of the inverse  $\sigma_k$  flow) that if the initial metric satisfies the Calabi symmetry, the flow converges to a Kähler current which is the sum of a Kähler metric with a conic singularity and a current of integration along a divisor.

We consider the case when  $X$  is a Kähler surface. As discussed above, a necessary and sufficient condition for convergence of the flow to a critical metric is:

$$(1.4) \quad [c\chi - \omega] > 0.$$

Donaldson [D1] remarked that if this condition fails, then one might expect the  $J$ -flow to blow up over some curves of negative self-intersection. It was observed in [SW] (see Proposition 4.5 there) that, applying the results of Buchdahl [Bu] and Lamari [La], there exist a finite number  $N \geq 0$ , say, of irreducible curves  $C_i$  with  $C_i^2 < 0$  on  $X$  and positive real numbers  $a_i$  such that  $[c\chi - \omega] - \sum_{i=1}^N a_i [C_i]$  is Kähler. Moreover, it was shown in [SW] that, at least for some sequence of points approaching some  $C_i$  that the quantity  $|\varphi| + |\Delta_\omega \varphi|$  blows up.

In this paper we describe the behavior of the  $J$ -flow for certain classes  $[\chi]$  on the boundary of  $\mathcal{C}_\omega$ . First we introduce some notation: given a closed (1,1)-form  $\alpha$ , write  $[\alpha] \geq 0$  if there exists a smooth closed nonnegative (1,1) form cohomologous to  $\alpha$ . We consider any Kähler class  $[\chi]$  satisfying:

$$(1.5) \quad [c\chi - \omega] \geq 0.$$

All such classes  $[\chi]$  lie in the closure of  $\mathcal{C}_\omega$ . However, the converse is false in general. Indeed the boundary of  $\mathcal{C}_\omega$  consists of Kähler classes  $[\chi]$  such that  $[c\chi - \omega]$  is *nef*, which means that it has nonnegative intersection with every curve in  $X$ . Further, since

$$[c\chi - \omega]^2 = [\omega]^2 > 0,$$

the class  $[c\chi - \omega]$  is nef and big. Nevertheless, it need not satisfy (1.5) - see Question 4.1 below. However, in many cases the condition (1.5) is equivalent to  $[\chi]$  belonging to the closure of  $\mathcal{C}_\omega$  in the Kähler cone. This holds for all Hirzebruch surfaces, for example, since explicit nonnegative (1,1) forms can be found representing all classes on the boundary of the Kähler cone (see the discussion in [Ca]).

Our main result is:

**Theorem 1.1** (Main Theorem). *Let  $X$  be a Kähler surface with Kähler metrics  $\omega$  and  $\chi$  such that  $[c\chi - \omega] \geq 0$ . Then there exist a finite number of curves  $C_i$  on  $X$  of*

negative self-intersection such that the solution  $\varphi(t)$  of the  $J$ -flow (1.1) converges in  $C_{\text{loc}}^\infty(X \setminus \bigcup C_i)$  to a continuous function  $\varphi_\infty$ , smooth on  $X \setminus \bigcup C_i$ , satisfying

$$(1.6) \quad c\chi_{\varphi_\infty}^2 = 2\chi_{\varphi_\infty} \wedge \omega, \quad \text{for } \chi_{\varphi_\infty} = \chi + dd^c \varphi_\infty \geq 0.$$

Moreover,  $\varphi_\infty$  is the unique continuous solution of (1.6) up to the addition of a constant.

Our result makes use of some recent works in the study of complex Monge-Ampère equations that appeared after the breakthrough of Kołodziej [K1]. We apply the  $C^0$  estimate of Eyssidieux-Guedj-Zeriahi [EGZ1] and Zhang [Z] for the degenerate complex Monge-Ampère equation. By comparing with this solution, we obtain our key uniform estimate for  $\varphi(t)$  along the  $J$ -flow (Proposition 2.2 below). In addition, we use the viscosity methods introduced in [EGZ2] to give a second proof of our key estimate. Moreover, the results of [EGZ2] allow us to conclude that the solution of (1.6) above is continuous, and also that (1.6) can be understood in both the pluripotential and the viscosity senses.

We have an application of our result to the *Mabuchi energy* [M], a functional which is closely connected to the problem of algebraic stability and existence of constant scalar curvature Kähler (cscK) metrics [Y2, T1, D2]. Given a Kähler surface  $(X, \chi)$ , the Mabuchi energy is the functional  $\text{Mab} : \mathcal{P}_\chi \rightarrow \mathbb{R}$  given by

$$\text{Mab}(\varphi) = - \int_0^1 \int_X \frac{\partial \varphi_t}{\partial t} (R_{\chi_{\varphi_t}} - \mu) \chi_{\varphi_t}^n dt,$$

where  $\{\varphi_t\}_{0 \leq t \leq 1}$  is a path in  $\mathcal{P}_\chi$  between 0 and  $\varphi$ ,  $R_{\chi_{\varphi_t}}$  is the scalar curvature of the metric  $\chi_{\varphi_t}$  and  $\mu$  is the average of the scalar curvature of  $\chi$ . The value  $\text{Mab}(\varphi)$  is independent of the choice of path.

It was conjectured by Tian [T1] that the existence of a cscK metric is equivalent to the *properness* of the Mabuchi energy, meaning that there exists an increasing function  $f : [0, \infty) \rightarrow \mathbb{R}$  with  $\lim_{x \rightarrow \infty} f(x) = \infty$  such that

$$\text{Mab}(\varphi) \geq f(E(\varphi)), \quad \text{where} \quad E(\varphi) = \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge (\chi_0 + \chi_\varphi).$$

This conjecture holds whenever  $c_1(X) < 0$ ,  $c_1(X) = 0$  or on manifolds with  $c_1(X) > 0$  if there are no holomorphic vector fields [T1, T2, TZ]. In fact in each case, the function  $f$  can be taken to be linear [T2, PSSW]. Chen [Ch1] showed that on manifolds with  $c_1(X) < 0$ , or equivalently, with ample canonical bundle  $K_X$ , the Mabuchi energy can be written as a sum of two terms: the first is the  $\mathcal{J}$ -functional with reference metric  $\omega$  in  $[K_X]$  and the second is a term which is bounded below. In fact the second term is proper [T2] (see the discussion in [SW]), and under the cohomological condition  $[c\chi - \omega] \geq 0$ , the  $\mathcal{J}$  functional has a lower bound, as shown in Proposition 3.3 below. Hence we obtain:

**Corollary 1.2.** *Suppose that  $X$  is a Kähler surface with ample canonical bundle  $K_X$ . Then the Mabuchi energy is proper on the classes  $[\chi]$  satisfying:*

$$(1.7) \quad \left( \frac{2[\chi] \cdot [K_X]}{[\chi]^2} \right) [\chi] - [K_X] \geq 0.$$

Moreover, the function  $f$  in the definition of properness can be taken to be linear.

Thus, conjecturally, classes  $[\chi]$  in the cone given by (1.7) should admit cscK metrics. Note that the class  $[K_X]$  is inside this cone and admits a cscK metric

[A, Y1]. The same is true for classes sufficiently close to  $[K_X]$  (see [LS]). On the other hand, Ross found Kähler classes on surfaces with  $K_X$  ample that do not admit cscK metrics [R]. Corollary 1.2, together with the arguments of [LS], suggests that the set of classes that admit cscK metrics is strictly larger than those lying in the cone (1.7).

An outline of the paper is as follows. In Section 2, we prove the key  $C^0$  estimate. We provide two proofs: the first uses smooth maximum principle arguments and the second uses the notion of viscosity solutions from [EGZ2]. We complete the proof of the main theorem in Section 3, and in the last section we finish with some questions for further study.

## 2. THE $C^0$ ESTIMATE

For convenience of notation we assume from now on that  $c = 1$ . We may do this because the condition  $[c\chi - \omega] \geq 0$  is invariant under scalings of  $[\chi]$ . In addition, we may assume, by modifying the initial data if necessary, that  $\chi - \omega \geq 0$ .

The key estimate we need is a uniform  $C^0$  estimate for the solution  $\varphi(t)$  of the  $J$ -flow. We need the following theorem on the degenerate complex Monge-Ampère equation [EGZ1, EGZ2] (the  $C^0$  estimate was proved independently in [Z] under slightly less general hypotheses).

**Theorem 2.1.** *Let  $(M, \omega)$  be a compact Kähler manifold of complex dimension  $n$  and let  $\alpha$  be a semi-positive  $(1, 1)$  form with  $\int_M \alpha^n > 0$ . For any nonnegative  $f \in L^p(M, \omega^n)$ , for  $p > 1$ , with  $\int_M f \omega^n = \int_M \alpha^n$  there exists a unique continuous function  $\varphi$  on  $M$  with  $\alpha + dd^c \varphi \geq 0$  and*

$$(2.1) \quad (\alpha + dd^c \varphi)^n = f \omega^n, \quad \sup_M \varphi = 0.$$

Moreover,  $\|\varphi\|_{C^0(M)}$  is uniformly bounded by a constant depending only on  $p, M, \omega, \alpha$  and  $\|f\|_{L^p(M)}$ .

Given this, we immediately obtain a solution  $\varphi_\infty$  to (1.6), using the observation of Chen [Ch1] that the critical equation can be rewritten as a complex Monge-Ampère equation:

$$(2.2) \quad \chi_\varphi^2 = 2\chi_\varphi \wedge \omega \iff (\chi_\varphi - \omega)^2 = \omega^2.$$

Writing  $\alpha := \chi - \omega \geq 0$  on the Kähler surface  $X$ , we can apply Theorem 2.1 to see that there exists a continuous function  $\varphi_\infty$  solving (1.6). Moreover,  $\varphi_\infty$  is unique up to the addition of a constant.

Next we use the uniform  $C^0$  bound from Theorem 2.1 to obtain:

**Proposition 2.2.** *Let  $\varphi(t)$  be the solution of  $J$ -flow (1.1) on the Kähler surface  $X$ . Then there exists  $C$  depending only on the initial data such that for all  $t \geq 0$ ,*

$$(2.3) \quad \|\varphi(t)\|_{C^0(X)} \leq C.$$

*Proof.* From the introduction, we know

$$[\chi - \omega] - \sum_{i=1}^N a_i [C_i] > 0,$$

for positive real numbers  $a_i$  and irreducible curves  $C_i$  of negative self-intersection. Moreover, since we are assuming  $[\chi - \omega] \geq 0$ , we may take the constants  $a_i$  to be arbitrarily small. However, we will not need to make use of this last fact.

It follows that there exist Hermitian metrics  $h_i$  on the line bundles  $[C_i]$  associated to  $C_i$  such that

$$(2.4) \quad \chi - \omega - \sum_{i=1}^N a_i R_{h_i} > 0,$$

where  $R_{h_i} = -dd^c \log h_i$  is the curvature of  $h_i$ . Let  $s_i$  be a holomorphic section of  $[C_i]$  vanishing along  $C_i$  to order 1. Recall that we denote  $\chi - \omega$  by  $\alpha$ .

Next, we apply Theorem 2.1 and write  $\psi$  for the solution to the degenerate complex Monge-Ampère equation

$$(2.5) \quad (\alpha + dd^c \psi)^2 = \omega^2, \quad \alpha + dd^c \psi \geq 0,$$

subject to the condition  $\sup_X \psi = 0$ . We have  $\|\psi\|_{C^0(X)} \leq C$ .

Moreover,  $\psi$  is smooth away from the curves  $C_i$ . Indeed this last statement follows from a trick of Tsuji [Ts], as used in [EGZ1]. Although the proof is the same, the precise statement we need does not seem to be quite contained in [EGZ1], and so we briefly outline the idea here for the convenience of the reader. For  $\delta > 0$ , let  $\psi_\delta$  be Yau's solution of the complex Monge-Ampère equation

$$(2.6) \quad (\alpha + \delta\omega + dd^c \psi_\delta)^2 = c_\delta \omega^2, \quad \alpha_\delta := \alpha + \delta\omega + dd^c \psi_\delta > 0,$$

for a constant  $c_\delta$  chosen so that the integrals of both sides are equal. From Theorem 2.1,  $\psi_\delta$  is uniformly bounded in  $C^0$ . To obtain a second order estimate for  $\psi_\delta$ , uniform in  $\delta$ , we consider, for a constant  $A > 0$ ,

$$(2.7) \quad Q_\delta = \log \operatorname{tr}_\omega \alpha_\delta - A(\psi_\delta - \sum_i a_i \log |s_i|_{h_i}^2),$$

which is well-defined on  $X \setminus \bigcup C_i$  and tends to  $-\infty$  on  $\bigcup C_i$ . Compute at a point in  $X \setminus C_i$ ,

$$\Delta_{\alpha_\delta} Q_\delta \geq -C \operatorname{tr}_{\alpha_\delta} \omega - 2A + A \operatorname{tr}_{\alpha_\delta} (\alpha - \sum_i a_i R_{h_i}).$$

Then using (2.4) we may choose  $A$  sufficiently large so that at a maximum of  $Q_\delta$ ,  $\operatorname{tr}_{\alpha_\delta} \omega$  is bounded from above. It then follows from (2.6) that  $Q_\delta$  is uniformly bounded from above. This establishes estimates, uniform in  $\delta$ , for  $dd^c \psi_\delta$  on compact subsets of  $X \setminus \bigcup C_i$ . The  $C_{\text{loc}}^\infty(X \setminus \bigcup C_i)$  estimates for  $\psi_\delta$  then follow from the standard local theory for the complex Monge-Ampère equation. Taking a limit as  $\delta \rightarrow 0$  shows that  $\psi$  is smooth away from the  $C_i$ .

Fix  $\varepsilon \in (0, 1)$ . We will apply the maximum principle to the quantity

$$\theta_\varepsilon = \varphi - (1 + \varepsilon)\psi + \varepsilon \sum_{i=1}^N a_i \log |s_i|_{h_i}^2 - A\varepsilon t,$$

where  $A$  is a constant to be determined. Observe that  $\theta_\varepsilon$  is smooth on  $X \setminus \bigcup C_i$  and tends to negative infinity along  $\bigcup C_i$  and hence for each time  $t$ ,  $\theta_\varepsilon$  achieves a maximum in the interior of  $X \setminus \bigcup C_i$ .

We rewrite (1.1) as follows

$$(2.8) \quad \begin{aligned} \frac{\partial \varphi}{\partial t} &= 1 - \frac{2\chi_\varphi \wedge \omega}{\chi_\varphi^2} = \frac{\chi_\varphi^2 - 2\chi_\varphi \wedge \omega}{\chi_\varphi^2} \\ &= \frac{(\chi_\varphi - \omega)^2 - \omega^2}{\chi_\varphi^2} = \frac{\omega^2}{\chi_\varphi^2} \left( \frac{(\chi_\varphi - \omega)^2}{\omega^2} - 1 \right) = \frac{\omega^2}{\chi_\varphi^2} \left( \frac{\alpha_\varphi^2}{\alpha_\psi^2} - 1 \right). \end{aligned}$$

Compute on  $X \setminus \bigcup C_i$ , using (2.8),

$$\begin{aligned} \frac{\partial}{\partial t} \theta_\varepsilon &= \frac{\omega^2}{\chi_\varphi^2} \left( \frac{(\alpha + dd^c \varphi)^2}{(\alpha + dd^c \psi)^2} - 1 \right) - A\varepsilon \\ &= \frac{\omega^2}{\chi_\varphi^2} \left( \frac{((1 + \varepsilon)\alpha + (1 + \varepsilon)dd^c \psi - \varepsilon(\alpha - \sum a_i R_{h_i}) + dd^c \theta_\varepsilon)^2}{(\alpha + dd^c \psi)^2} - 1 \right) - A\varepsilon. \end{aligned}$$

But  $\alpha - \sum a_i R_{h_i} \geq 0$ , and at the maximum of  $\theta_\varepsilon$ , we have  $dd^c \theta_\varepsilon \leq 0$ . Hence at the maximum of  $\theta_\varepsilon$ ,

$$(2.9) \quad \frac{\partial}{\partial t} \theta_\varepsilon \leq \frac{\omega^2}{\chi_\varphi^2} \left( (1 + \varepsilon)^2 \frac{(\alpha + dd^c \psi)^2}{(\alpha + dd^c \psi)^2} - 1 \right) - A\varepsilon < 0,$$

if we choose

$$A = \sup_{X \times [0, \infty)} \frac{3\omega^2}{\chi_\varphi^2},$$

which is a uniform constant since  $\chi_\varphi$  is always uniformly bounded from below away from zero along the  $J$ -flow. Indeed this last fact follows immediately from taking a time derivative of the  $J$ -flow equation and applying the maximum principle (see Lemma 4.1 in [Ch2]). Then (2.9) implies that  $\theta_\varepsilon$  must achieve its maximum at time zero, and hence  $\theta_\varepsilon$  is uniformly bounded from above by a constant independent of  $\varepsilon$ . Letting  $\varepsilon \rightarrow 0$  we obtain the upper bound for  $\varphi$ .

The lower bound of  $\varphi$  is similar: just replace  $\varepsilon$  by  $-\varepsilon$  and consider the minimum instead of the maximum.  $\square$

We provide a second proof. The proof is based on the equivalence of two notions of weak solution of (2.2): pluripotential sense and viscosity sense.

*Second proof of Proposition 2.2.* As in the first proof, write  $\psi$  for the solution to (2.5) with  $\sup_X \psi = 0$ . The function  $\psi$  is continuous on  $X$  and is smooth away from the curves  $C_i$ . We now apply Theorem 3.6 of [EGZ2], which states that  $\psi$  satisfies the equation (2.5) in the viscosity sense as defined in that paper.

We refer to [EGZ2] for the precise definition of a viscosity solution to (2.5). Instead we state two consequences of this definition which are sufficient for our purposes:

- (i) If  $x_0$  is any point on  $X$  and  $q$  is any smooth function defined in a neighborhood of  $x_0$  such that

$$\psi - q \text{ has a local maximum at } x_0$$

then  $(\alpha + dd^c q)^2 \geq \omega^2$  at  $x_0$ .

- (ii) If  $x_0$  is any point on  $X$  and  $q$  is any smooth function defined in a neighborhood of  $x_0$  such that

$$\psi - q \text{ has a local minimum at } x_0$$

then  $(\alpha + dd^c q)^2 \leq \omega^2$  at  $x_0$ .

Indeed (i) follows from the definition of viscosity subsolution and (ii) from the definition of viscosity supersolution (see Section 2 in [EGZ2]).

We first find an upper bound for  $\varphi$ . Let  $\varepsilon > 0$  and define  $H_\varepsilon = \varphi - \psi - \varepsilon t$ . We wish to show that  $H_\varepsilon$  attains its maximum value at  $t = 0$ . Note that  $H_\varepsilon$  satisfies

the equation:

$$\frac{\partial H_\varepsilon}{\partial t} = 1 - \frac{2\chi_\varphi \wedge \omega}{\chi_\varphi^2} - \varepsilon.$$

Suppose that  $H_\varepsilon$  attains a maximum at a point  $(x_0, t_0)$  on  $X \times [0, T]$  for some finite  $T > 0$ , and assume for a contradiction that  $t_0 > 0$ . Then  $\partial H_\varepsilon / \partial t(x_0, t_0) \geq 0$ . Define a smooth function  $q$  on  $X$  by  $q(x) = \varphi(x, t_0) - H_\varepsilon(x_0, t_0) - \varepsilon t_0$ . Note that the function

$$x \mapsto (\psi - q)(x) = -H_\varepsilon(x, t_0) + H_\varepsilon(x_0, t_0)$$

achieves its minimum at  $x_0$ . Then we can apply (ii) to see that  $(\alpha + dd^c q)^2 \leq \omega^2$  at  $x_0$ , or in other words

$$(\chi - \omega + dd^c \varphi)^2 \leq \omega^2, \quad \text{at } (x_0, t_0),$$

which is equivalent to

$$\chi_\varphi^2 \leq 2\chi_\varphi \wedge \omega \quad \text{at } (x_0, t_0).$$

It follows that

$$\frac{\partial H_\varepsilon}{\partial t}(x_0, t_0) = 1 - \frac{2\chi_\varphi \wedge \omega}{\chi_\varphi^2} - \varepsilon < 0,$$

contradicting the fact that  $\partial H_\varepsilon / \partial t(x_0, t_0) \geq 0$ . Hence  $H_\varepsilon$  attains its maximum value at  $t = 0$  and is uniformly bounded from above independent of  $\varepsilon$ . Letting  $\varepsilon \rightarrow 0$  gives the desired upper bound for  $\varphi$ .

Applying a similar argument, using (i) instead of (ii), gives a uniform lower bound for  $\varphi$ .  $\square$

We can now apply Theorem 1.3 of [SW] together with the standard local theory for (1.1) to obtain:

**Proposition 2.3.** *For any compact subset  $K \subset X \setminus \bigcup C_i$  and any  $k \geq 0$ , there exists a constant  $C_{k,K}$  such that for all  $t$ ,*

$$\|\varphi(t)\|_{C^k(K)} \leq C_{k,K}.$$

### 3. PROOF OF THE MAIN THEOREM

Again we assume in this section that  $[\chi]$  is scaled so that  $c = 1$ . Before proving the main theorem we first discuss the  $\mathcal{J}$  and  $\mathcal{I}$ -functionals. Define  $\mathcal{J}_{\omega,\chi}$  and  $\mathcal{I}_{\omega,\chi}$  by

$$\begin{aligned} \mathcal{J}_{\omega,\chi}(\varphi) &:= \int_0^1 \int_X \dot{\varphi}_t (2\chi_{\varphi_t} \wedge \omega - \chi_{\varphi_t}^2) dt, \\ \mathcal{I}_{\omega,\chi}(\varphi) &:= \int_0^1 \int_X \dot{\varphi}_t \chi_{\varphi_t}^2 dt, \end{aligned}$$

where  $\varphi_t$  is a smooth path in  $\mathcal{P}_\chi$  connecting 0 and  $\varphi$ . For simplicity, we will omit the subscripts.

Note that if  $\varphi(t)$  is the solution of the  $J$ -flow, then

$$(3.1) \quad \frac{d}{dt} \mathcal{J}(\varphi(t)) = - \int_X \dot{\varphi}(t)^2 \chi_{\varphi(t)}^2, \quad \frac{d}{dt} \mathcal{I}(\varphi(t)) = 0.$$

In particular, the  $J$ -flow is the gradient flow of  $\mathcal{J}$ .

One can write explicit formulae for  $\mathcal{J}$ ,  $\mathcal{I}$  as follows:

$$(3.2) \quad \mathcal{J}(\varphi) = \int_X \varphi(\chi_\varphi \wedge \omega + \chi \wedge \omega) - \frac{1}{3} \int_X \varphi(\chi_\varphi^2 + \chi_\varphi \wedge \chi + \chi^2),$$

$$(3.3) \quad \mathcal{I}(\varphi) = \frac{1}{3} \int_X \varphi(\chi_\varphi^2 + \chi_\varphi \wedge \chi + \chi^2).$$

Thus an immediate corollary of Proposition 2.2 is:

**Proposition 3.1.** *There exists a uniform constant  $C$  such that, for  $\varphi(t)$  the solution of the  $J$ -flow, we have*

$$\mathcal{J}(\varphi(t)) \geq -C$$

for all  $t \geq 0$ .

In what follows, we will need to make use of a simple continuity-type result for the  $\mathcal{I}$  and  $\mathcal{J}$  functionals.

**Lemma 3.2.** *Let  $\varphi_j \in \mathcal{P}_\chi$  and let  $\varphi$  be a continuous function on  $X$  satisfying  $\chi + dd^c \varphi \geq 0$ . Let  $Y$  be a proper subvariety of  $X$ . Suppose that*

- (a) *there exists  $C$  such that  $\|\varphi_j\|_{C^0(X)} \leq C$ .*
- (b)  *$\varphi_j \rightarrow \varphi$  in  $C_{\text{loc}}^\infty(X \setminus Y)$  as  $j \rightarrow \infty$ .*

Then

$$\mathcal{J}(\varphi_j) \rightarrow \mathcal{J}(\varphi) \quad \text{and} \quad \mathcal{I}(\varphi_j) \rightarrow \mathcal{I}(\varphi) \quad \text{as } j \rightarrow \infty.$$

*Proof.* The proof is a simple exercise in pluripotential theory (we refer the reader to [K2] for an introduction to this theory). For the convenience of the reader we sketch here the proof. Note first that for  $\varphi$  continuous with  $\chi + dd^c \varphi \geq 0$ , the quantities  $\chi_\varphi^2$ ,  $\chi \wedge \chi_\varphi$  and  $\chi_\varphi \wedge \omega$  define finite measures on  $X$  and hence from (3.2) and (3.3) the functionals  $\mathcal{I}(\varphi)$  and  $\mathcal{J}(\varphi)$  are well-defined.

We may choose a sequence of open tubular neighborhoods  $Y_k$  of  $Y$  such that  $Y_k \downarrow Y$  as  $k \rightarrow \infty$ . Since  $Y$  is pluripolar, the capacity  $\text{Cap}_\chi(Y)$  of  $Y$  with respect to  $\chi$  (in the sense of Kolodziej [K1]) is zero. By the properties of this capacity (see [GZ] for example) we have

$$\lim_{k \rightarrow \infty} \text{Cap}_\chi(Y_k) = \text{Cap}_\chi(Y) = 0.$$

Since the  $\varphi_j$  are uniformly bounded, it follows that  $\int_{Y_k} \varphi_j \beta \wedge \gamma \rightarrow 0$  as  $k \rightarrow \infty$ , uniformly in  $j$ , where  $\beta, \gamma$  are each one of  $\omega, \chi$  or  $\chi_{\varphi_j}$ . The same holds if we replace  $\varphi_j$  by  $\varphi$ . The result then follows from the expressions (3.2) and (3.3) together with condition (b).  $\square$

Finally we give the proof of the main theorem.

*Proof of Theorem 1.1.* Since  $\mathcal{J}$  is decreasing and bounded from below, there exists a constant  $C$  such that

$$(3.4) \quad \int_0^\infty \int_X \dot{\varphi}(t)^2 \chi_{\varphi(t)}^2 dt < C.$$

We claim that for each fixed point  $p \in X \setminus \bigcup C_i$ ,  $\dot{\varphi}(p, t) \rightarrow 0$  as  $t \rightarrow \infty$ . Indeed, suppose not. Then there exists  $\varepsilon > 0$  and a sequence of times  $t_i \rightarrow \infty$  such that  $|\dot{\varphi}(t_i)| > \varepsilon$  for all  $i$ . But since we have bounds for  $\dot{\varphi}$  and all its time and space derivatives in a fixed neighborhood  $U$ , say, of  $p$  with  $U \subset X \setminus \bigcup C_i$ , it follows that



$|\dot{\varphi}(t)| > \varepsilon/2$  for  $t \in [t_i, t_i + \delta]$  for a uniform  $\delta > 0$ . This contradicts (3.4) and establishes the claim.

Since we have  $C_{\text{loc}}^\infty(X \setminus \bigcup C_i)$  bounds for  $\dot{\varphi}$ , the uniqueness of limits implies that  $\dot{\varphi}$  converges to zero in  $C_{\text{loc}}^\infty(X \setminus \bigcup C_i)$ .

We have uniform  $C^\infty$  bounds for  $\varphi(t)$  on compact subsets of  $X \setminus \bigcup C_i$ , and hence we can apply the Arzela-Ascoli theorem to see that for a sequence of times  $t_i \rightarrow \infty$  we have  $\varphi(t_i) \rightarrow \varphi_\infty$  for a smooth (bounded) function  $\varphi_\infty$  on  $X \setminus \bigcup C_i$ . Moreover, since  $\dot{\varphi} \rightarrow 0$ ,  $\varphi_\infty$  satisfies the equation  $\chi_{\varphi_\infty}^2 = 2\chi_{\varphi_\infty} \wedge \omega$  as in the statement of the theorem.

We also have  $\mathcal{I}(\varphi_\infty) = \lim_{t \rightarrow \infty} \mathcal{I}(\varphi(t)) = \mathcal{I}(\varphi_0)$ , using Lemma 3.2 and the fact that  $\mathcal{I}$  is constant along the flow. Applying Theorem 2.1, we know that (1.6) has a unique solution up to the addition of a constant. Thus  $\varphi_\infty$  is the unique solution of (1.6) subject to the condition  $\mathcal{I}(\varphi_\infty) = \mathcal{I}(\varphi_0)$ .

Finally we claim that  $\varphi(t)$  converges in  $C_{\text{loc}}^\infty(X \setminus \bigcup C_i)$  to  $\varphi_\infty$ . Suppose not. Then there exists  $\varepsilon > 0$  and a sequence of times  $t_i \rightarrow \infty$  such that  $\|\varphi(t_i) - \varphi_\infty\|_{C^k(K)} > \varepsilon$  for all  $i$ , for some integer  $k$  and compact  $K \subset X \setminus \bigcup C_i$ . Since we have uniform  $C^\infty$  bounds for  $\varphi(t)$  on  $K$  we can pass to a subsequence and assume that  $\varphi(t_i)$  converges to a function  $\varphi'_\infty \neq \varphi_\infty$ . But  $\varphi'_\infty$  will also satisfy the equation  $\chi_{\varphi'_\infty}^2 = 2\chi_{\varphi'_\infty} \wedge \omega$  and  $\mathcal{I}(\varphi'_\infty) = \mathcal{I}(\varphi_0)$  contradicting the uniqueness.  $\square$

As a consequence:

**Corollary 3.3.** *The  $\mathcal{J}$ -functional is bounded from below on  $\mathcal{P}_\chi$ .*

*Proof.* Take any  $\varphi_0 \in \mathcal{P}_\chi$ . Then running the  $J$ -flow from  $\varphi_0$ , which by Theorem 1.1 converges to  $\varphi_\infty$ , we obtain (applying Lemma 3.2),

$$\mathcal{J}(\varphi_0) \geq \lim_{t \rightarrow \infty} \mathcal{J}(\varphi(t)) = \mathcal{J}(\varphi_\infty),$$

since  $\mathcal{J}$  is decreasing along the flow.  $\square$

Finally:

*Proof of Corollary 1.2.* Combine Corollary 3.3 and Lemma 4.1 of [SW].  $\square$

#### 4. FURTHER QUESTIONS

Finally, we propose some further questions.

*Question 4.1.* In general, a nef and big class that is not Kähler cannot be represented by a smooth nonnegative  $(1,1)$  form (see Example 1.7 of [DPS]). so what if we assume only that  $[\chi - \omega]$  is nef and big? In this case, by [BEGZ], we know that we can produce a solution  $\psi$  of (2.2) with very mild singularities along  $C_i$  (less than any log pole). Can it be translated into an estimate for the solution  $\varphi(t)$  of the  $J$ -flow? Does it imply that the  $\mathcal{J}$ -functional is bounded from below?

*Question 4.2.* The results of [FL2] indicate a possible picture when  $[\chi]$  is outside of  $\mathcal{C}_\omega$ . But they assume both  $\omega$  and  $\chi$  are of Calabi ansatz. Can one prove a general result on Kähler surfaces? In this case, presumably the  $\mathcal{J}$ -functional is not bounded from below.

*Question 4.3.* For general  $n$ , it would be interesting to investigate the weak solution of the critical equation (1.2) when  $[\chi]$  does not lie in  $\mathcal{C}_\omega$ .

## REFERENCES

- [A] Aubin, T., *Équations du type Monge-Ampère sur les variétés kählériennes compactes*, Bull. Sci. Math. (2) 102 (1978), no. 1, 63–95
- [Bu] Buchdahl, N., *On compact Kähler surfaces*, Ann. Inst. Fourier (Grenoble) 49, no. 1 (1999), 287–302
- [BEGZ] Boucksom, S., Eyssidieux, P., Guedj, V. and Zeriahi, A., *Monge-Ampère equations in big cohomology classes*, Acta Math. 205 (2010), no.2, 199–262
- [Ca] Calabi, E. *Extremal Kähler metrics*, in Seminar on Differential Geometry, pp. 259–290, Ann. of Math. Stud. 102, Princeton Univ. Press, Princeton, N.J., 1982
- [Ch1] Chen, X., *On the lower bound of the Mabuchi energy and its application*, Int. Math. Res. Notices 12 (2000), 607–623
- [Ch2] Chen, X., *A new parabolic flow in Kähler manifolds*, Comm. Anal. Geom. 12, no.4 (2004), 837–852
- [D1] Donaldson, S.K., *Moment maps and diffeomorphisms*, Asian J. Math. 3, no. 1 (1999), 1–16
- [D2] Donaldson, S.K. *Scalar curvature and stability of toric varieties*, J. Differential Geom. 62 (2002), 289–349
- [DPS] Demailly, J.-P., Peternell, T. and Schneider, M., *Compact complex manifolds with numerically effective tangent bundles*, J. Algebraic Geom. 3, (1994) 295–345
- [EGZ1] Eyssidieux, P., Guedj, V. and Zeriahi, A., *Singular Kähler-Einstein metrics*, J. Amer. Math. Soc. 22 (2009), 607–639
- [EGZ2] Eyssidieux, P., Guedj, V. and Zeriahi, A., *Viscosity solutions to degenerate complex Monge-Ampère equations*, 64, (2011) 1059–1094
- [FL1] Fang, H. and Lai, M., *On the geometric flows solving Kählerian inverse  $\sigma_k$  equations*, arXiv:1203.2571, to appear in Pac. J. Math
- [FL2] Fang, H. and Lai, M., *Convergence of general inverse  $\sigma_k$ -flow on Kähler manifolds with Calabi Ansatz*, arXiv:1203.5253
- [FLM] Fang, H., Lai, M. and Ma, X., *On a class of fully nonlinear flows in Kähler geometry*, J. reine angew. Math. 653 (2011), 189–220
- [GZ] Guedj, V. and Zeriahi, A. *Intrinsic capacities on compact Kähler manifolds*, J. Geom. Anal. 15 (2005), no. 4, 607–639
- [K1] Kolodziej, S. *The complex Monge-Ampère equation*, Acta Math. 180 (1998), no. 1, 69–117
- [K2] Kolodziej, S. *The complex Monge-Ampère equation and pluripotential theory*, Mem. Amer. Math. Soc. 178 (2005), no. 840, x+64 pp
- [La] Lamari, A., *Le cône Kählérien d’une surface*, J. Math. Pures. Appl. 78 (1999), 249–263
- [LS] LeBrun, C. and Simanca, S. R. *Extremal Kähler metrics and complex deformation theory*, Geom. Funct. Anal. 4 (1994), no. 3, 298–336
- [M] Mabuchi, T., *K-energy maps integrating Futaki invariants*, Tohoku Math. J., 38, 1986, 245–257
- [PSSW] Phong, D.H., Song, J., Sturm, J. and Weinkove, B., *The Moser-Trudinger inequality on Kähler-Einstein manifolds*, Amer. J. Math. 130 (2008), no. 4, 1067–1085
- [R] Ross, J. *Unstable products of smooth curves*, Invent. Math. 165 (2006), no. 1, 153–162
- [SW] Song, J. and Weinkove, B., *The convergence and singularities of the J-flow with applications to the Mabuchi energy*, Comm. Pure Appl. Math. 61 (2008), no. 2, 210–229
- [T1] Tian, G. *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. 130 (1997), no. 1, 1–37
- [T2] Tian, G. *Canonical metrics in Kähler geometry*. Notes taken by Meike Akveld. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2000
- [TZ] Tian, G. and Zhu, X. *A nonlinear inequality of Moser-Trudinger type*, Calc. Var. Partial Differential Equations 10 (2000), no. 4, 349–354
- [Ts] Tsuji, H. *Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type*, Math. Ann. 281 (1988), no. 1, 123–133
- [W1] Weinkove, B., *Convergence of the J-flow on Kähler surfaces*, Comm. Anal. Geom. 12, no. 4(2004), 949–965
- [W2] Weinkove, B., *On the J-flow in higher dimensions and the lower boundedness of the Mabuchi energy*, J. Differential Geom. 73 (2006), no. 2, 351–358
- [Y1] Yau, S.-T., *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*, Comm. Pure Appl. Math. 31 (1978), 339–411

- [Y2] Yau, S.-T., *Open problems in geometry*, Proc. Symposia Pure Math. 54 (1993), 1–28
- [Z] Zhang, Z., *On degenerate Monge-Ampère equations over closed Kähler manifolds*, Int. Math. Res. Not. 2006, Art. ID 63640, 18pp

HAO FANG, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOWA  
*E-mail address:* `hao-fang@uiowa.edu`

MIJIA LAI, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROCHESTER  
*E-mail address:* `lai@math.rochester.edu`

JIAN SONG, DEPARTMENT OF MATHEMATICS, RUTGERS  
*E-mail address:* `jiansong@math.rutgers.edu`

BEN WEINKOVE, MATHEMATICS DEPARTMENT, UNIVERSITY OF CALIFORNIA, SAN DIEGO  
*E-mail address:* `weinkove@math.ucsd.edu`